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SUPERCUSPIDAL REPRESENTATIONS AND POINCARÉ SERIES OVER FUNCTION FIELDS

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ABSTRACT. In this paper, we will give a new construction of certain cusp forms on $GL(2)$ over a rational function field. The forms which we construct are analogs of holomorphic modular forms, in that the local representations at the infinite place are in the discrete series. The novelty of our approach is that we are able to give a very explicit construction of these forms as certain 'Poincaré series.' We will also study the exponential sums which arise in the Fourier expansions of these Poincaré series.

In this paper, we will give a new construction of certain cusp forms on $GL(2)$ over a rational function field. The forms which we construct are analogs of holomorphic modular forms, in that the local representations at the infinite place are in the discrete series. The novelty of our approach is that we are able to give a very explicit construction of these forms as certain 'Poincaré series.' We will also study the exponential sums which arise in the Fourier expansions of these Poincaré series. The particular forms which we will construct are lifts from $GL(1)$ over the unique nonramified quadratic extension of the global field which arises from extension of the finite field of constants, and indeed, they have been considered before by Gekeler [1]. We expect that the Poincaré series construction will apply to general automorphic forms over function fields with a supercuspidal representation at infinity. The construction is also valid over the metaplectic group. However, we shall not consider these topics here.

Let E/F be a quadratic extension of global fields. Let A_F and A_E be the corresponding adele rings, and let ν be a Grössencharacter of E. As usual, we may regard ν as a character of the idele class group $A_E^\times/E^\times$ of E, which is a subgroup of index two in the relative Weil group $W_{E/F}$. By Jacquet and Langlands [2, Proposition 12.1], if ν does not factor through the norm map $A_E^\times \to A_F^\times$, the two dimensional representation of $W_{E/F}$ induced from ν corresponds to an automorphic cuspidal representation $\pi(\nu)$ of $GL(2, A_F)$.

Let F_* be a finite field of odd cardinality q, and let E_* be its unique quadratic extension. Let $F = F_*(T)$ and $E = E_*(T)$ be the corresponding fields of rational functions in one variable T. A special role will be played by the two places of this field corresponding to the points $T = 0$ and $T = \infty$ on the rational curve, and by abuse of notation we will denote these two places as 0 and ∞, respectively. Let ν_* be a character of $E_*^\times$. We will make the assumption that ν_* is trivial on $F_*^\times$ but nontrivial on the kernel of the norm

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map $E_*^\times \to F_*^\times$. We may describe a Grössencharacter of E as follows. Let $R_E = E_*[T]$ be the ring of polynomials. We will describe a character of ideals in R_E which are prime to T. As usual, such a character may be associated with a character of the idele class group, and by a common abuse of notation, we will use the same letter ν to denote these two characters, of ideals and of idele classes. An ideal $\mathfrak{a}$ of R_E which is prime to T has a unique generator $A(T) \in E_*[T]$ which is monic; we define

$$\nu(\mathfrak{a}) = (-1)^{\deg A}\,\nu_*(A(0)).$$

This Hecke character has the property that its restriction to F is the quadratic character of the extension E/F, and consequently the automorphic representation $\pi(\nu)$ will have trivial central character; that is, will be an automorphic representation of $G(A_F)$, where G denotes the algebraic group $PGL(2)$.

The usual construction of $\pi(\nu)$ is by means of the Weil representation. However, we will give an alternative construction by means of a Poincaré series. We will associate with ν a supercuspidal representation ρ of $GL(2, F_*((t)))$, where t is an indeterminate and $F_*((t))$ is the field of formal power series. We will prove that there exists a unique automorphic representation of $GL(2, A_F)$ which is unramified at every place except $T = 0$ and $T = \infty$, and which is isomorphic to ρ at these places under the specializations $t = T$ and $t = T^{-1}$. The method of proof of this theorem is entirely elementary, based on Mackey theory. The proof also leads directly to the construction of the Poincaré series. Since the cuspidal automorphic representation $\pi(\nu)$ constructed by means of the Weil representation agrees with this characterization, we see that the Poincaré series must lie in the subspace $\pi(\nu)$ of $L^2(G_F\backslash G_A)$.

We will consider the problem of directly computing the Fourier coefficients of these Poincaré series. The eigenvalues of the Hecke operators are known, because of the identification of the corresponding automorphic representation as $\pi(\nu)$. However, the proof that this Poincaré series corresponds to $\pi(\nu)$, which is contained in the proof of Theorems 1.3 and 1.4, in contradistinction to the entirely elementary proofs of Theorems 1.1 and 1.2, depends upon some machinery. Although we thus know these eigenvalues in advance, it is nevertheless of interest to us to give a direct and elementary calculation of the Fourier coefficients which does not make use of the Weil representation. We will succeed in directly evaluating the Hecke eigenvalues indexed by primes of degree ≤ 4. This leads to some very interesting exponential sum calculations.

The result that the Poincaré series which we construct may be also constructed by means of the Weil representation is also a consequence of Gekeler [1, Satz 2.2]. That paper contains a rather thorough discussion of automorphic forms of small conductor over rational function fields. He shows there are no cusp forms with conductor a divisor of degree < 4, and he classifies the cusp forms of conductor a divisor of degree precisely 4. (In the case which we consider here, the conductor is the divisor $2 \cdot (0) + 2 \cdot (\infty)$.) As a result of this classification, he is able to show that all cusp forms with conductor of degree four come from the Weil representation. Although the particular automorphic forms which we construct in this paper have thus been previously considered, nevertheless their construction here as Poincaré series appears to be new. This construction will work for automorphic forms over function fields whenever the infinite component(s) are supercuspidal. Moreover, the construction is also valid over the metaplectic group.

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1. A supercuspidal representation and a Poincaré series

We will consider functions on both $GL(2)$ and $PGL(2)$, and it will be convenient to use different notations for matrices to distinguish the two groups. Thus we will denote by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

an element of $GL(2)$, and the corresponding coset in $PGL(2)$, respectively.

Kutzko [3] gave definitive results on the construction of supercuspidal representations of $GL(2)$ over a local field as induced representations. Although our Theorem 1.1 is contained in the results of Kutzko, we will prove what we need from scratch to emphasize the elementary nature of the result, and the symmetry between the proofs of Theorems 1 and 2, both of which are based on Mackey theory.

First let E_∞/F_∞ be a nonramified quadratic extension of nonarchimedean local fields, and let E_*/F_* be the corresponding quadratic extension of residue fields. Let ν_* be a character of $E_*^\times$ which is nontrivial on the kernel of the norm map $E_*^\times \to F_*^\times$. Following Piatetski-Shapiro [5] there exists an irreducible representation $\rho_* : GL(2, F_*) \to \mathrm{End}_{\mathbb{C}}(V_*)$ whose character χ_* satisfies

$$\chi_* \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} q-1 & \text{if } x = 0; \\ -1 & \text{otherwise;} \end{cases}$$

$\chi_*(g) = 0$ if g is hyperbolic; and $\chi_*(g) = -\nu_*(\alpha) - \nu_*(\beta)$ if g is elliptic with eigenvalues α and β in $E_* - F_*$. This representation is *cuspidal* in the sense that there exists no vector in V_* which is invariant under a maximal unipotent subgroup, or equivalently, that there exists no linear form L on V_* such that

$$L\left(\rho_* \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} v\right) = L(v)$$

for $x \in F_*$, $v \in V_*$. We will assume that ν_* is trivial on $F_*^\times$. Consequently the central character of ρ_* is trivial, and we may regard ρ_* as a representation of $G(F_*)$, where $G = PGL(2)$.

Let $\mathscr{O}_\infty$ be ring of integers in F_∞, so that $G(\mathscr{O}_\infty)$ is a maximal compact subgroup of $G(F_\infty)$. The canonical map $\mathscr{O}_\infty \to F_*$ induces a homomorphism $G(\mathscr{O}_\infty) \to G(F_*)$. Let ρ_∞ denote the pullback of ρ_* to $G(\mathscr{O}_\infty)$ under this homomorphism. Let ρ be the representation of $G(F_\infty)$ obtained by compact induction. Thus ρ acts on the space V_ρ of compactly supported functions $f : G(F_\infty) \to V_*$ which satisfy

$$f(kg) = \rho_\infty(k) f(g), \qquad k \in G(\mathscr{O}_\infty),\ g \in G(F_\infty).$$

The action is by right translation:

$$(\rho(g)f)(g') = f(g'g).$$

We recall that a smooth representation r of $G(F_\infty)$ on a space V is *supercuspidal* if there exists no linear form L on V such that

$$L\left(r\begin{bmatrix}1 & x\\ & 1\end{bmatrix}v\right) = L(v)$$

for $x \in F_\infty$, $v \in V$.

Theorem 1.1. *The representation* ρ *constructed above is an irreducible supercuspidal representation.*

Proof. Clearly ρ is a smooth representation. We will use Mackey theory to show that it is irreducible. Since ρ is a subrepresentation of $L^2(G(F_\infty)) \otimes V_*$, it is unitary, and so by the converse to Schur's Lemma, it is sufficient to show that $\mathrm{Hom}_{G(F_\infty)}(V_\rho, V_\rho)$ is one dimensional. Let $L : V_\rho \to V_\rho$ be an intertwining operator. If $g \in G(F_\infty)$ and $v \in V_*$, define an element of V_ρ by

$$f_{g,v}(h) = \begin{cases} \rho_\infty(k)v & \text{if } h = kg,\ k \in G(\mathcal{O}_\infty); \\ 0 & \text{if } h \notin G(\mathcal{O}_\infty)g. \end{cases}$$

Any element of V_ρ is a finite linear combination of functions of this type. It is easily verified that if g, $g' \in G(F_\infty)$, $k \in G(\mathcal{O}_\infty)$, $v \in V_*$, then

$$f_{kg,\,\rho_\infty(k)v} = f_{g,v}, \qquad \rho(g')\, f_{gg',v} = f_{g,v},$$

and if $F \in V_\rho$,

$$F = \sum_{\gamma \in G(\mathcal{O}_\infty)\backslash G(F_\infty)} f_{\gamma, F(\gamma)}.$$

Define a function $\Delta : G(F_\infty) \to \mathrm{End}_{\mathbb{C}}(V_*)$ by

$$\Delta(g)\, v = L(f_{g^{-1},v})(e),$$

where e is the identity element of $G(F_\infty)$. Then it follows from the above-mentioned properties of $f_{g,v}$ that for k, $k' \in G(\mathcal{O}_\infty)$, we have

$$\Delta(kgk') = \rho_\infty(k) \circ \Delta(g) \circ \rho_\infty(k'). \tag{1.1}$$

The intertwining operator L may be reconstructed from Δ because for $f \in V_\rho$,

$$(Lf)(g) = \sum_{\gamma \in G(\mathcal{O}_\infty)\backslash G(F_\infty)} \Delta(\gamma^{-1})\, f(\gamma g).$$

Hence it is sufficient to show that the space of functions Δ satisfying (1.1) is one dimensional.

First let us show that (1.1) implies that the function Δ is supported on $G(\mathcal{O}_\infty)$. By the p-adic Cartan decomposition, a complete set of double coset representatives for $G(\mathcal{O}_\infty)\backslash G(F_\infty)/G(\mathcal{O}_\infty)$ consists of the matrices

$$t_n = \begin{bmatrix}1 & \\ & \varpi^n\end{bmatrix},$$

where $n \geq 0$. Here ϖ is a local uniformizing parameter. We will show that $\Delta(t_n) = 0$ unless $n = 0$. If $n > 0$, then consider that

$$\begin{bmatrix}1 & x\\ & 1\end{bmatrix} t_n = t_n \begin{bmatrix}1 & \varpi^n x\\ & 1\end{bmatrix}.$$

Let $x \in \mathscr{O}_\infty$, and let $\overline{x}$ denote the image of $x \in F_*$, and let

$$n = \begin{bmatrix} 1 & \overline{x} \\ & 1 \end{bmatrix} \in G(F_*). \tag{1.2}$$

Then (1.1) implies that $\rho_\infty(n) \circ \Delta(t_n) = \Delta(t_n)$. Hence for $v \in V_*$, $\Delta(t_n)v$ is fixed by the group of upper triangular unipotent elements of $G(F_*)$, and since ρ_* is cuspidal, this implies that $\Delta(t_n)v = 0$. Therefore Δ is supported on $G(\mathscr{O}_\infty)$, and Δ is completely determined by the value of $\Delta(e)$. But by (1.1), $\Delta(e)$ commutes with the action of $G(\mathscr{O}_\infty)$, and so by Schur's Lemma is a scalar endomorphism of V_*. This proves that the space of Δ satisfying (1.1) is one dimensional, and hence that ρ is irreducible.

Now we prove that ρ is supercuspidal. It is sufficient to show that if T is any linear functional on V_ρ such that

$$T\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} f\right) = T(f),$$

then $T = 0$. It is sufficient to show that $T(f_{g,v}) = 0$. If $k \in G(\mathscr{O}_\infty)$, we have $f_{kg,\rho_\infty(k)v} = f_{g,v}$, so we may assume that

$$g = \begin{bmatrix} \varpi^n & \lambda \\ & 1 \end{bmatrix}.$$

For this g, we have, with $x \in \mathscr{O}_\infty$,

$$\rho \begin{bmatrix} 1 & \varpi^{-n}x \\ & 1 \end{bmatrix} f_{g,v} = f_{g,\rho_\infty(n)v},$$

where n is given by (1.2). Hence $T(f_{g,v}) = T(f_{g,\rho_*(n)v})$. Since ρ_* is cuspidal, this implies that $T(f_{g,v}) = 0$. Hence ρ is supercuspidal. □

We will now assume that $F = F_*(T)$ and $E = E_*(T)$ as in the Introduction. If v is any place of F, we will denote by F_v the completion of F at v, and by $\mathscr{O}_v$ the ring of integers in F_v, so in particular $F_\infty = F_*((T^{-1}))$, $F_0 = F_*((T))$, $\mathscr{O}_\infty = F_*[[T^{-1}]]$ and $\mathscr{O}_0 = F_*[[T]]$. Also let $E_\infty = E_*((T^{-1}))$. Let Γ be the group $G(R_F)$, where $R_F = F_*[T]$. Then Γ is a discrete subgroup of $G(F_\infty)$, and the quotient has finite volume. We may ask whether ρ occurs in $L^2(\Gamma\backslash G(F_\infty))$, but it will follow from Theorem 1.2 below that it does not. We ask therefore a slightly more general question. Let $\sigma_* : G(F_*) \to \mathrm{End}_{\mathbb{C}}(U_*)$ be another irreducible representation of this finite group. The residue map $R_F \to F_*$ determined by $T \mapsto 0$ induces a homomorphism $\Gamma \to G(F_*)$. Let σ_Γ denote the pullback of σ_* to Γ under this homomorphism. Let $L^2(\Gamma\backslash G(F_\infty), \sigma_\Gamma)$ be the space of functions $\phi : G(F_\infty) \to U_*$ which satisfy $\phi(\gamma g) = \sigma_\Gamma(\gamma)\,\phi(g)$, and which are square integrable modulo Γ.

Theorem 1.2. *The dimension of*

$$\mathrm{Hom}_{G(F_\infty)}(\rho, L^2(\Gamma\backslash G(F_\infty), \sigma_\Gamma))$$

equals one if $\sigma_* \cong \rho_*$, and zero otherwise.

Proof. Since both ρ and $L^2(\Gamma\backslash G(F_\infty), \sigma_\Gamma)$ are induced representations, we may use Mackey theory to consider this question. If $v \in V_*$, $g \in G(F_\infty)$, let $f_{g,v}$ be the element of $\rho = \mathrm{ind}_{G(\mathscr{O}_\infty)}^{G(F_\infty)}(\rho_\infty)$ which was defined in the proof of Theorem 1.1. Suppose that $L : \rho \to L^2(\Gamma\backslash G(F_\infty), \sigma_\Gamma)$ is an intertwining operator. We define a function $\Delta : G(F_\infty) \to \mathrm{Hom}_{\mathbb{C}}(V_*, U_*)$ as follows. Let $\Delta(g)v = L(f_{g^{-1},v})(e)$. It may be checked that for $k \in G(\mathscr{O}_\infty)$, $g \in G(F_\infty)$ and $\gamma \in \Gamma$ we have

$$\Delta(\gamma g k) = \sigma_\Gamma(\gamma) \circ \Delta(g) \circ \rho_\infty(k). \tag{1.3}$$

Now let us determine which double cosets of $\Gamma\backslash G(F_\infty)/G(\mathscr{O}_\infty)$ can support a function satisfying (1.3). A full set of representatives for these double cosets are the matrices

$$\begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix}, \qquad n \geq 0.$$

Suppose that $n > 0$. Then for $\alpha \in F_*$,

$$\begin{bmatrix} 1 & T^n\alpha \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} = \begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ & 1 \end{bmatrix}.$$

Hence (1.3) implies that

$$\Delta\begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} \circ \rho_\infty \begin{bmatrix} 1 & \alpha \\ & 1 \end{bmatrix} = \Delta\begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix},$$

and since ρ_* is cuspidal, this implies that

$$\Delta\begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} = 0.$$

This shows that Δ is supported on the single double coset $n = 0$, and is determined by $\Delta(e)$. It follows from (1.3) that $\Delta(e)$ is an intertwining operator for the representations ρ_* and σ_* of $PGL(2, F_*) = G(\mathscr{O}_\infty) \cap \Gamma$, and as such, can exist only if $\rho_* \cong \sigma_*$, in which case the space of possible functions Δ is one dimensional by Schur's Lemma. □

Theorem 1.2 therefore motivates us to take $\sigma_* = \rho_*$ and to examine the space $\mathscr{P}(\rho)$ of automorphic forms which is the image of the unique homomorphism $\rho \to L^2(\Gamma\backslash G(F_\infty), \rho_\Gamma)$, where ρ_Γ denotes the pullback of ρ_* to Γ. We are interested in knowing the eigenvalues of the Hecke operators on elements of this space. To define the Hecke operators, let $\phi \in \mathscr{P}(\rho)$. Let P be a monic, irreducible polynomial in R_F other than T. Let

$$(\mathscr{T}_P\phi)(g) = \rho_*\begin{bmatrix} P(0)^{-1} & \\ & 1 \end{bmatrix} \phi\left(\begin{bmatrix} P & \\ & 1 \end{bmatrix} g\right) + \rho_*\begin{bmatrix} 1 & \\ & P(0)^{-1} \end{bmatrix} \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & bT \\ & P \end{bmatrix} g\right).$$

We will prove that

$$\mathscr{T}_P\phi = q^{\deg(P)/2}\lambda_P\,\phi \tag{1.4}$$

for some eigenvalue $\lambda_P \in \mathbb{C}$. Let L be an element of

$$\mathrm{Hom}_{G(F_\infty)}(\rho, L^2(\Gamma\backslash G(F_\infty), \sigma_\Gamma)),$$

unique up to constant multiple by Theorem 1.2. Consider the composition $T_p \circ L$. Then this too is an element of the one-dimensional space of Theorem 1.2, hence is a constant multiple of L. Since $\mathscr{P}(\rho)$ is defined to be the image of L, this proves the existence of a constant such that (1.4) is satisfied.

The proof of Theorem 1.2 leads to an explicit construction of an automorphic form $\phi \in \mathscr{P}(\rho)$ by means of a Poincaré series. If $f \in V_\rho$, the image of f under the intertwining operator L is given by the formula

$$L(f)(g) = \sum_{G(\mathscr{O}_\infty)\backslash G(F_\infty)} \Delta(\gamma^{-1})\, f(\gamma g).$$

Taking into account that Δ is supported on $\Gamma G(\mathscr{O}_\infty)$, this may be written as

$$L(f)(g) = \sum_{\Gamma\cap G(\mathscr{O}_\infty)\backslash\Gamma} \Delta(\gamma^{-1})\, f(\gamma g) = \sum_{\Gamma\cap G(\mathscr{O}_\infty)\backslash\Gamma} \rho_\Gamma(\gamma^{-1})\, f(\gamma g). \tag{1.5}$$

To obtain ϕ, we apply this to the function

$$f(g) = \begin{cases} \rho_\infty(g)\, v_0 & \text{if } g \in G(\mathscr{O}_\infty); \\ 0 & \text{otherwise,} \end{cases}$$

where $v_0 \in V_*$ is the unique vector which is invariant under $\rho_*(g)$ for all diagonal $g \in G(F_*)$. Then $\phi = L(f)$ is given by the Poincaré series (1.5).

Before we compute the eigenvalues λ_P, we need an adelic form of Theorem 1.2. It is now necessary to distinguish between representations of the completions of F at 0 and ∞. As in Theorems 1.1 and 1.2, let ρ be the representation of $G(F_\infty)$ induced compactly from the representation ρ_∞ of $G(\mathscr{O}_\infty)$, which is the representation ρ_* lifted to $G(\mathscr{O}_\infty)$ via $T^{-1} \mapsto 0$, and similarly, let $\rho' : G(F_0) \to \mathrm{End}(V_{\rho'})$ be the representation induced from ρ_0, which is the same representation ρ_* lifted to $G(\mathscr{O}_0)$ via $T \mapsto 0$.

Theorem 1.3. *There is at exactly one automorphic representation π of $G(A_F)$ which is in the unramified principal series at each place except at 0 and ∞, and whose components $\pi_\infty \cong \rho$ and $\pi_0 \cong \rho'$.*

Proof. Let us show first that

$$G(F)\, G(F_\infty) \prod_{v\neq\infty} G(\mathscr{O}_v) = G(A). \tag{1.6}$$

Indeed, we consider the space of R_F-lattices in F_∞^2. Each lattice is determined by its localizations, so $G(A)$ acts on this space. The stabilizer of the standard lattice R_F^2 is $G(F_\infty) \prod_{v\neq\infty} G(\mathscr{O}_v)$, and so we may identify the space of lattices with the space of cosets $G(A)/G(F_\infty) \prod_{v\neq\infty} G(\mathscr{O}_v)$. Now since R_F is a principal ideal domain, the action of $G(F)$ on lattices is transitive, whence (1.6).

We will now construct an isomorphism

$$\begin{aligned} \alpha : {} & L^2(\Gamma\backslash G(F_\infty), \rho_\Gamma) \\ & \to \mathrm{Hom}_{G(F_0)}\left(V_{\rho'},\, L^2\left(G(F)\backslash G(A)/ \prod_{v\neq 0,\infty} G(\mathscr{O}_v)\right)\right) \end{aligned} \tag{1.7}$$

as $G(F_\infty)$-modules. To see this, we first define, for $\phi \in L^2(\Gamma\backslash G(F_\infty), \rho_\Gamma)$, a function $\Lambda\phi : G(A) \to V_*$. Indeed, if $g \in G(A)$, by (1.6) we may write

$g = g_F g_\infty \gamma_0 \gamma_m$ where $g_F \in G(F)$, $g_\infty \in G(F_\infty)$, $\gamma_0 \in G(\mathscr{O}_0)$ and $\gamma_m \in \prod_{v \neq 0, \infty} G(\mathscr{O}_v)$. The decomposition is not unique, but nevertheless

$$(\Lambda\phi)(g) = \rho_0(\gamma_0)^{-1} \phi(g_\infty)$$

is well defined. We may now define the mapping (1.7) by

$$(\alpha(\phi)(f))(g) = \int_{G(F_0)} \langle (\Lambda\phi)(g g_0^{-1}), f(g_0) \rangle \, dg_0,$$

for $\phi \in L^2(\Gamma \backslash G(F_\infty), \rho_\Gamma)$, $f \in V_{\rho'}$, $g \in G(A)$, where $\langle\ ,\ \rangle$ is an ρ_*-invariant inner product on V_*. It be checked that this is an isomorphism.

Now the isomorphism (1.7) and Theorem 1.2 imply that

$$\mathrm{Hom}_{G(F_\infty)\times G(F_0)}\left(V_\rho \otimes V_{\rho'}, L^2\left(G(F)\backslash G(A) / \prod_{v\neq 0, \infty} G(\mathscr{O}_v) \right)\right)$$

is one dimensional. Recalling that a nonramified principal series representation of $G(F_v)$ for $v \neq 0, \infty$ has a unique $G(\mathscr{O}_v)$-fixed vector, this is equivalent to Theorem 1.3. □

We may now determine the eigenvalues λ_P.

Theorem 1.4. *If the degree d of P is odd, then $\lambda_P = 0$. If d is even, then we may factor $P = P_1 P_2$ in R_E, with P_1 and P_2 monic. We have*

$$\lambda_P = (-1)^{d/2} [\nu_*(P_1(0)) + \nu_*(P_2(0))].$$

Proof. It will be convenient to adopt the adele language for the proof of this theorem. Let $\pi = \pi(\nu)$ be the automorphic representation described in the introduction. It is easy to see that the Hecke eigenvalues associated with π are as described in this theorem, and so it is sufficient to show that π agrees with the automorphic representation of Theorem 1.3. Evidently π is unramified except at 0 and ∞, and so it is sufficient to show that $\pi_\infty \cong \rho$ and $\pi_0 \cong \rho'$. Let us assume for definiteness that $v = \infty$—the other case is clearly identical. By Frobenius reciprocity,

$$\mathrm{Hom}_{G(F_\infty)}(\rho, \pi_\infty) \cong \mathrm{Hom}_{G(\mathscr{O}_\infty)}(\rho_\infty, \pi_\infty),$$

and we will show that this space of intertwining operators is nonempty. This will prove that $\rho \cong \pi_\infty$, as required.

The local component ν_∞ is then the representation of $E_\infty = E_*((T^{-1}))$ which agrees with the lift of ν_* to the local units, and such that $\nu_\infty(T^{-1}) = -1$. Recall that π is constructed by means of the Weil representation, and by Propositions 12.1, 4.6 and 1.5 of [2], the local component π_∞ has the following description. We will denote by $x \mapsto \overline{x}$ the Galois automorphism of E_∞ over F_∞, so that $\overline{x} = x^q$ if $x \in E_*$. Let ψ_* be an additive character of F_*, and let ψ be the additive character of F given by

$$\psi\left(\sum a_k T^{-k}\right) = \psi_*(a_0), \qquad a_i \in F_*.$$

The group $G(F_\infty)$ has a subgroup G^+ of index two consisting of cosets of the center in $GL(2, F_\infty)$ with representatives g such that $\det(g)$ is a norm from E_∞. (Since E_∞/F_∞ is unramified, this will be the case if and only if the valuation of $\det(g)$ is even.) Then π_∞ is induced from a representation r of G^+ on the space V_r of locally constant compactly supported functions Φ on E_∞ which satisfy $\Phi(xy) = \nu(y)^{-1}$ when $y\overline{y} = 1$. The representation r is given on generators of G^+ by the following prescription:

$$\left(r\begin{bmatrix} 1 & y \\ & 1 \end{bmatrix}\Phi\right)(x) = \psi(x\overline{x}.y)\,\Phi(x),$$

$$\left(r\begin{bmatrix} u & \\ & u^{-1} \end{bmatrix}\Phi\right)(x) = |u|_E^{1/2}\,\Phi(ux),$$

$$\left(r\begin{bmatrix} u & \\ & 1 \end{bmatrix}\Phi\right)(x) = |v|_E^{1/2}\,\nu(v)^{-1}\Phi(vx),$$

where v is chosen so that $v\overline{v} = u$, and

$$\left(r\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\Phi\right)(x) = -\hat{\Phi}(x),$$

where the Fourier transform

$$\hat{\Phi}(x) = \int_E \Phi(xy)\,\psi(x\overline{y} + y\overline{x})\,dx.$$

The measure on E is normalized so that $\mathscr{O}_\infty(E) = E_*[[T^{-1}]]$ has volume one. Note that $G(\mathscr{O}_\infty) \subset G^+$. By Frobenius reciprocity, it is sufficient to construct a nonzero $G(\mathscr{O}_\infty)$ intertwining operator $V_* \to V_r$. Now we have a similar description of the representation ρ_* of the finite field. We may interpret ρ_* as acting on the space of functions on E_* which satisfy $\Phi(xy) = \nu(y)^{-1}$ when $y\overline{y} = 1$. In this realization, ρ_* acts on generators of G^+ by

$$\left(\rho_*\begin{bmatrix} 1 & y \\ & 1 \end{bmatrix}\Phi\right)(x) = \psi(x\overline{x}.y)\,\Phi(x),$$

$$\left(\rho_*\begin{bmatrix} u & \\ & u^{-1} \end{bmatrix}\Phi\right)(x) = \Phi(ux),$$

$$\left(\rho_*\begin{bmatrix} u & \\ & 1 \end{bmatrix}\Phi\right)(x) = \nu(v)^{-1}\Phi(vx),$$

where v is chosen so that $v\overline{v} = u$, and

$$\left(\rho_*\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\Phi\right)(x) = -\hat{\Phi}(x),$$

where the finite Fourier transform

$$\hat{\Phi}(x) = \frac{1}{q}\sum_{E_*}\Phi(xy)\,\psi(x\overline{y} + y\overline{x})\,dx.$$

Now a nonzero $G(\mathscr{O}_\infty)$ intertwining operator $V_* \to V_r$ may be obtained by simply lifting a function on E_* to $\mathscr{O}_E$, and extending it by zero to E. This completes the proof of Theorem 1.3. □

2. Evaluation of the eigenvalues of the Hecke operators

In this section, we evaluate $\mathscr{T}_P\phi$ for a specific ϕ to obtain the eigenvalue λ_P of $\mathscr{T}_P$ where P is an irreducible polynomial whose degree is 2, 4, or odd. This evaluation will be independent of the proof of Theorem 1.4. Define ϕ_0 and ϕ as follows: for $g \in G(F_\infty)$, letting

$$g = \gamma \begin{bmatrix} T^n & \\ & 1 \end{bmatrix} k$$

$$\phi_0(g) = \begin{cases} \chi_*(\gamma(0)k(\infty)) & \text{if } n = 0; \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi(g) = \frac{1}{q-1} \sum_u \phi_0\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix} g\right).$$

We observe that while this scalar-valued function ϕ is not the same as the vector valued automorphic form which was denoted ϕ in the previous section, it is obtained from that function by composition with a linear functional on the vector space V_*. (This does not effect the Hecke eigenvalues, of course.) To see this, we observe that V_* contains a vector v_0, unique up to a constant, such that

$$\rho_*\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix}\right) v_0 = v_0$$

for all $u \in F_*^\times$, and also that there exists a linear functional H on V_*, also unique up to a constant such that

$$H\left(\rho_*\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix}\right) v\right) = H(v)$$

for all $u \in F_*^\times$ and all $v \in V$. Consequently $g \mapsto H(\rho_*(g)v_0)$ is the unique element in the ring of matrix coefficients for the representation ρ_* which is bi-invariant under the diagonal subgroup, whence, after multiplying H by a suitable constant, we have

$$H(\rho_*(g)v_0) = \frac{1}{q-1} \sum_{u \in F_*^\times} \chi_*\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix} g\right)$$

for $g = G(F_*)$. This shows that the function ϕ considered in this section is simply the vector-valued function (1.5) considered in §1 composed with the linear functional H.

We would like to evaluate

$$\mathcal{T}_P\phi\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} P & \\ & 1 \end{bmatrix}\right) + \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix}\right).$$

Then,

$$\lambda_P = q^{-d/2} \mathcal{T}_P\phi\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right),$$

where d is the degree of P. Since

$$\begin{bmatrix} P & \\ & 1 \end{bmatrix} = \begin{bmatrix} T^d & \\ & 1 \end{bmatrix} \begin{bmatrix} P/T^d & \\ & 1 \end{bmatrix}, \quad \phi\left(\begin{bmatrix} P & \\ & 1 \end{bmatrix}\right) = 0.$$

Thus

$$\mathcal{T}_P\phi\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right) = \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix}\right).$$

Let us call the right-hand side of the above equation S_P. First, for polynomials of odd degree, we will prove

Theorem 2.1. *If* $\deg(P)$ *is odd, then* $S_P = 0$ *and hence* $\lambda_P = 0$.

Proof. It is sufficient to show that for any g in $G(F_\infty)$, if $\operatorname{ord}(\det g)$ is odd, then $\phi_0(g) = 0$. Let

$$g = \gamma \begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} k.$$

Since $\det \gamma$ and $\det k$ are units, $\operatorname{ord}(\det \gamma)$ and $\operatorname{ord}(\det k)$ are both 0. Hence, if $\operatorname{ord}(\det g)$ is odd, then n cannot be zero. Thus $\phi_0(g) = 0$ by the definition. □

To evaluate S_P for polynomials of degree 2 or 4, we need several lemmas.

Lemma 2.2. *For any* g *in* $G(F_\infty)$ *and* k *in* $G(\mathcal{O}_\infty)$,

$$\phi_0(gk) = \phi_0(gk(\infty)) = \phi_0(k(\infty)g).$$

Proof. By the definition of ϕ_0 and the fact that χ_* is a character. □

Lemma 2.3. *For any* g *in* $G(F_\infty)$,

$$\sum_{v \neq 0, w} \phi\left(g \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \psi_*(vw) = q\phi(g).$$

Proof.

$$\begin{aligned} &\sum_{v \neq 0, w} \phi\left(g \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \psi_*(vw) \\ &\quad = \sum_{v, w} \phi\left(g \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \psi_*(vw) - \sum_{w} \phi\left(g \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \end{aligned}$$

which, by the cuspidality of ϕ,

$$= \sum_{v, w} \phi\left(g \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \psi_*(vw),$$

and since if $w \neq 0$, then $\sum_v \psi_*(vw) = 0$, this equals $q\phi(g)$. □

Lemma 2.4. *For any* g *in* $G(F_\infty)$ *and* v *in* $F_*^\times$,

$$\phi\left(g \begin{bmatrix} v & \\ & 1 \end{bmatrix}\right) = \phi(g).$$

Proof. By the definition of ϕ and Lemma 2.2. □

Lemma 2.5. *For any* g *in* $G(F_\infty)$,

$$\sum_{w} \phi\left(\begin{bmatrix} 1 & w \\ & 1 \end{bmatrix} g \begin{bmatrix} 1 & -w \\ & 1 \end{bmatrix}\right) = \frac{q}{q-1}\phi_0(g).$$

Proof. Define $S(u, g)$ as

$$S(u, g) = \sum_{w} \phi_0\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix} g \begin{bmatrix} 1 & -w \\ & 1 \end{bmatrix}\right).$$

It is enough to show

$$S(u, g) = \begin{cases} q\phi_0(g) & \text{if } u = 1; \\ 0 & \text{otherwise.} \end{cases}$$

If $u = 1$, using Lemma 2.2,

$$\begin{aligned} S(u, g) &= \sum_w \phi_0\left(\begin{bmatrix} 1 & w \\ & 1 \end{bmatrix} g \begin{bmatrix} 1 & -w \\ & 1 \end{bmatrix}\right) \\ &= \sum_w \phi_0\left(\begin{bmatrix} 1 & -w \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix} g\right) \\ &= q\phi_0(g). \end{aligned}$$

If $u \neq 1$, using the cuspidality of ϕ_0,

$$\begin{aligned} S(u, g) &= \sum_w \phi_0\left(\begin{bmatrix} 1 & w \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & (u-1)w \\ & 1 \end{bmatrix} \begin{bmatrix} u & \\ & 1 \end{bmatrix} g \begin{bmatrix} 1 & -w \\ & 1 \end{bmatrix}\right) \\ &= \sum_w \phi_0\left(\begin{bmatrix} 1 & (u-1)w \\ & 1 \end{bmatrix} \begin{bmatrix} u & \\ & 1 \end{bmatrix} g\right) \\ &= 0. \quad \square \end{aligned}$$

Lemma 2.6. *Let P and Q be polynomials in R_F such that* $\deg(Q) < \deg(P)$. *Define B_m, D_m, E_m and F_m by*

$$B_1 = Q, \quad D_1 = P, \quad D_m = B_m E_m + F_m,$$
$$\deg(F_m) < \deg(B_m), \quad D_{m+1} = B_m, \quad B_{m+1} = F_m.$$

Then, for $\gamma \in \Gamma$ and $k \in G(\mathscr{O}_\infty)$,

$$\phi_0\left(\gamma \begin{bmatrix} 1 & Q \\ & P \end{bmatrix} k\right) = \begin{cases} \chi_*(\delta) & \textit{if } \deg(B_r) = (1/2)\deg(P); \\ 0 & \textit{otherwise,} \end{cases}$$

where r is the unique m such that $\deg(B_{m-1}) > (1/2)\deg(P) \geq \deg(B_m)$ *(here, B_0 is conventionally defined to be P), e_m is the 0th coefficient of E_m, b_r is the highest coefficient of B_r, and*

$$\delta = \gamma(0) \begin{bmatrix} & 1 \\ 1 & e_1 \end{bmatrix} \cdots \begin{bmatrix} & 1 \\ 1 & e_r \end{bmatrix} \begin{bmatrix} (-1)^r/b_r & \\ & b_r \end{bmatrix} k(\infty).$$

Proof. Since $\deg(B_m)$ is strictly decreasing, r in the statement of the lemma exists. Define A_m, C_m, and M_m as

$$M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1 & Q \\ & P \end{bmatrix},$$
$$M_{m+1} = \begin{bmatrix} A_{m+1} & B_{m+1} \\ C_{m+1} & D_{m+1} \end{bmatrix} = \begin{bmatrix} -E_m & 1 \\ 1 & \end{bmatrix} M_m.$$

It is easy to see the following three facts:

$$\det(M_m) = (-1)^{m-1} P, \tag{2.6.1}$$
$$\deg(C_m) < \deg(A_m), \tag{2.6.2}$$
$$\deg(A_r) < \frac{1}{2}\deg(P). \tag{2.6.3}$$

Now we consider two cases: $\deg(B_r) > \deg(A_r)$ and $\deg(B_r) \leq \deg(A_r)$.

In the first case, let $n = \deg(P) - 2\deg(B_r)$. Using (2.6.1), we obtain

$$\gamma \begin{bmatrix} 1 & Q \\ & P \end{bmatrix} k = \gamma_1 \begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} k_1,$$

where

$$\gamma_1 = \gamma \begin{bmatrix} & 1 \\ 1 & E_1 \end{bmatrix} \cdots \begin{bmatrix} & 1 \\ 1 & E_r \end{bmatrix}$$

and

$$k_1 = \begin{bmatrix} (-1)^r P/B_r T^{n+\deg(B_r)} & F_r/T^{n+\deg(B_r)} \\ & B_r/T^{\deg(B_r)} \end{bmatrix} \begin{bmatrix} 1 & \\ A_r/B_r & 1 \end{bmatrix} k.$$

It is clear that $\gamma_1 \in \Gamma$ and that k_1 is in $G(\mathcal{O}_\infty)$. Moreover, $n \geq 0$, and $n = 0$ if and only if $\deg(B_r) = (1/2)\deg(P)$. Now, applying the definition of ϕ_0, we obtain the statement of the lemma.

In the second case, let $n = \deg(P) - 2\deg(A_r)$. Using (2.6.1), we obtain

$$\gamma \begin{bmatrix} 1 & Q \\ & P \end{bmatrix} k = \gamma_2 \begin{bmatrix} 1 & \\ & T^{-n} \end{bmatrix} k_2,$$

where

$$\gamma_2 = \gamma \begin{bmatrix} & 1 \\ 1 & E_1 \end{bmatrix} \cdots \begin{bmatrix} & 1 \\ 1 & E_{r-1} \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

and

$$k_2 = \begin{bmatrix} (-1)^{r+1} P/A_r T^{n+\deg(A_r)} & C_r/T^{n+\deg(A_r)} \\ & A_r/T^{\deg(A_r)} \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & B_r/A_r \end{bmatrix} k.$$

By (2.6.2) and (2.6.3), we see that $n > 0$, $\gamma_2 \in \Gamma$, and $k_2 \in G(\mathcal{O}_\infty)$. Moreover, $\deg(B_r) \neq (1/2)\deg(P)$. Now, the statement of the lemma easily follows. □

Lemma 2.7. *Let* $P(T) = \sum_{i=0}^{d} \lambda_i T^i$ *and* $b = \sum_{i=0}^{d-1} a_i T^i$. *Then*

$$S_P = q \sum_{\deg(b) \leq d-2,\, monic} \phi_0\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-2} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix} \right).$$

Proof. First we prove

$$S_P = \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-1} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \right) \tag{2.7.1}$$

evaluating the Pth Fourier coefficient of ϕ in two different ways. That is, we prove

$$\begin{aligned} &\int_{TF_*[T]\backslash F_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix} \right) \psi(Px)\, dx \\ &= \frac{1}{(q-1)q^d} \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix} \right), \end{aligned} \tag{2.7.2}$$

$$\begin{aligned} &\int_{TF_*[T]\backslash F_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix} \right) \psi(Px)\, dx \\ &= \frac{1}{(q-1)q^d} \sum_{b \bmod P} \phi\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-1} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \right). \end{aligned} \tag{2.7.3}$$

The proofs of (2.7.2) and (2.7.3) are structurally very similar. So we will provide a detailed proof for (2.7.2) and an indication for the proof of (2.7.3).

$$\begin{aligned}
&\int_{TF_*[T]\backslash F_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix}\right)\psi(Px)\,dx \\
&\quad = \int_{\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix}\right)\psi(Px)\,dx \\
&\quad = \frac{1}{q^d}\int_{T^d\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & x/P \\ & 1 \end{bmatrix}\begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix}\right)\psi(x)\,dx \\
&\quad = \frac{1}{q^d}\int_{T^d\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & P \end{bmatrix}\begin{bmatrix} PT^{-d} & \\ & 1 \end{bmatrix}\right)\psi(x)\,dx \\
&\quad = \frac{1}{q^d}\int_{T^d\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & P \end{bmatrix}\right)\psi(x)\,dx \\
&\quad = \frac{1}{q^d}\sum_{\deg(b)\le d}\int_{T^{-1}\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & b+x \\ & P \end{bmatrix}\right)\psi(b+x)\,dx \\
&\quad = \frac{1}{q^d}\sum_{\deg(b)\le d}\int_{T^{-1}\mathscr{O}_\infty} \phi\left(\begin{bmatrix} 1 & b \\ & P \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right)\psi(b+x)\,dx
\end{aligned}$$

which, by Lemma 2.2, since x is in $T^{-1}\mathscr{O}_\infty$,

$$= \frac{1}{q^{d+1}}\sum_{\deg(b)\le d} \phi\left(\begin{bmatrix} 1 & b \\ & P \end{bmatrix}\right)\psi(b)$$

Now, define

$$S(v) = \sum_{\deg(b)\le d} \phi\left(\begin{bmatrix} 1 & b \\ & P \end{bmatrix}\right)\psi(vb)$$

for v in $F_*^\times$. We need to show

$$S(1) = \frac{q}{q-1}\sum_{\deg(b)\le d-1} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix}\right).$$

First notice,

$$\begin{aligned}
S(v) &= \sum_{\deg(b)\le d} \phi\left(\begin{bmatrix} v & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ & P \end{bmatrix}\begin{bmatrix} 1/v & \\ & 1 \end{bmatrix}\right)\psi(vb) \\
&= \sum_{\deg(b)\le d} \phi\left(\begin{bmatrix} 1 & vb \\ & P \end{bmatrix}\right)\psi(vb) \\
&= \sum_{\deg(b)\le d} \phi\left(\begin{bmatrix} 1 & b \\ & P \end{bmatrix}\right)\psi(b) \\
&= S(1).
\end{aligned}$$

Hence

$$S(1) = \frac{1}{q-1} \sum_{v \neq 0} S(v)$$
$$= \frac{1}{q-1} \sum_{\deg(b) \leq d-1} \sum_{v \neq 0, w} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix} \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}\right) \psi_*(vw)$$

which, by Lemma 2.3,

$$= \frac{q}{q-1} \sum_{\deg(b) \leq d-1} \phi\left(\begin{bmatrix} 1 & Tb \\ & P \end{bmatrix}\right)$$

This completes the proof of (2.7.2).

For the proof of (2.7.3), first we obtain

$$\int_{TF_*[T] \backslash F_\infty} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} T^{-d} & \\ & 1 \end{bmatrix}\right) \psi(Px)\, dx$$
$$= \frac{1}{q^{d+1}} \sum_{\deg(b) \leq d} \phi\left(\begin{bmatrix} 1 & b \\ & T^d \end{bmatrix}\right) \psi(bPT^{-d})$$

Then, the rest of the proof goes similarly to the one of (2.7.2). This ends the proof of (2.7.1). Now,

$$S_P = \sum_{\deg(b) \leq d-1} \phi\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-1} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix}\right)$$
$$= \sum_{\deg(b) \leq d-2} \sum_{a_{d-1}} \phi\left(\begin{bmatrix} 1 & a_{d-1} \\ & 1 \end{bmatrix} M \begin{bmatrix} 1 & -a_{d-1} \\ & 1 \end{bmatrix} \begin{bmatrix} 1/\lambda_0 & \\ & 1 \end{bmatrix}\right)$$

where

$$M = \begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-2} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}$$

and which, by Lemmas 2.4 and 2.5,

$$= \frac{q}{q-1} \sum_{\deg(b) \leq d-2} \phi_0\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-2} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}\right)$$

It is easy to see that if $b = vb'$ for v in $F_*^\times$,

$$\phi_0\left(\begin{bmatrix} 1 & Tb \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-2} a_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}\right)$$
$$= \phi_0\left(\begin{bmatrix} 1 & Tb' \\ & T^d \end{bmatrix} \begin{bmatrix} 1 & -\sum_{i=0}^{d-2} a'_i \lambda_{d-(i+1)} \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}\right)$$

This entails the statement of the lemma. □

Now, for polynomials of degree 2, we have

Theorem 2.8. *Let $P(T) = T^2 + \lambda_1 T + \lambda_0$ be an irreducible polynomial over R_F, and $P(T) = P_1(T)P_2(T)$ be the factorization of $P(T)$ over R_E. Then*

$$S_P = -q(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

Hence,

$$\lambda_P = -(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

Proof. Using Lemmas 2.6 and 2.7, $S_P = q\chi_*(M)$ where

$$M = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_1 \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}.$$

Since $\det(M) = \lambda_0$, $\mathrm{tr}(M) = \lambda_1$, and M cannot be the identity, M is conjugate to

$$\begin{bmatrix} & -\lambda_0 \\ 1 & \lambda_1 \end{bmatrix}$$

which is elliptic since $P(T)$ is irreducible. Thus, using the known values of the character χ_*, which were described at the beginning of §1,

$$\chi_*(M) = -\nu_*(P_1(0)) - \nu_*(P_2(0)).$$

Thus,

$$S_P = q\chi_*(M) = -q(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

□

For polynomials of degree 4, we need one additional lemma concerned with the evaluation of an exponential sum.

Lemma 2.9. *Let $P(T) = T^4 + \lambda_3 T^3 + \lambda_2 T^2 + \lambda_1 T + \lambda_0$ be an irreducible polynomial over R_F, and $P(T) = P_1(T)P_2(T)$ be the factorization of $P(T)$ over R_E. Then,*

$$\sum_{a_1^2 \neq a_0} \psi_*\left(\frac{a_0^2 + \lambda_0 + a_1(\lambda_3 a_0 + \lambda_2 a_1 + \lambda_1)}{a_1^2 - a_0}\right) = -q\psi_*(\mu) - \sum_u \psi_*(u^2 + c)$$

where $\mu = P_1(0) + P_2(0)$ and $c = (4\lambda_2 - \lambda_3^2)/4$.

Proof. Let us call the left-hand side of the equation of the lemma S. In the evaluation of S, it is critical that μ can also be realized as the unique root of the cubic resolvent $R(T)$ of $P(T)$. Here

$$R(T) = T^3 - \lambda_2 T^2 + (\lambda_1\lambda_3 - 4\lambda_0)T - \lambda_0\lambda_3^2 + 4\lambda_0\lambda_2 - \lambda_1^2.$$

Solving

$$\frac{a_0^2 + \lambda_0 + a_1(\lambda_3 a_0 + \lambda_2 a_1 + \lambda_1)}{a_1^2 - a_0} = y$$

for a_0, we have

$$a_0^2 + (a_1\lambda_3 + y)a_0 - ya_1^2 + a_1(\lambda_2 a_1 + \lambda_1) + \lambda_0 = 0.$$

The discriminant D of this equation is

$$\begin{aligned} D &= (a_1\lambda_3 + y)^2 + 4ya_1^2 - 4a_1(\lambda_2 a_1 + \lambda_1) - 4\lambda_0 \\ &= (y - c)(2a_1)^2 + (\lambda_3 y - 2\lambda_1)2a_1 + y^2 - 4\lambda_0. \end{aligned}$$

Let σ be the quadratic character of F_*. Then

$$S = \sum_{a_1, y} (1 + \sigma(D))\psi_*(y) = \sum_{a_1, y} \sigma(D)\psi_*(y)$$

which changing $2a_1$ to a_1,

$$= \sum_{a_1, y} \sigma((y-c)a_1^2 + (\lambda_3 y - 2\lambda_1)a_1 + y^2 - 4\lambda_0)\psi_*(y)$$
$$= \sum_{a_1, y\neq c} \sigma((y-c)a_1^2 + (\lambda_3 y - 2\lambda_1)a_1 + y^2 - 4\lambda_0)\psi_*(y)$$
$$+ \sum_{a_1} \sigma((\lambda_3 c - 2\lambda_1)a_1 + c^2 - 4\lambda_0)\psi_*(c).$$

Let S_1 and S_2 be the first and second terms of the last formula. In order to conclude the lemma, we will prove

$$\text{(2.9.1)} \qquad S_1 = \begin{cases} -q\psi_*(\mu) - \sum_u \psi_*(u^2+c) & \text{if } \mu \neq c, \\ -\sum_u \psi_*(u^2+c) & \text{otherwise;} \end{cases}$$
$$\text{(2.9.2)} \qquad S_2 = \begin{cases} -q\psi_*(\mu) & \text{if } \mu = c, \\ 0 & \text{otherwise.} \end{cases}$$

First, for (2.9.1),

$$S_1 = \sum_{a_1, y\neq c} \sigma\left((y-c)\left(a_1 + \frac{\lambda_3 y - 2\lambda_1}{2(y-c)}\right)^2 + \frac{R(y)}{y-c}\right)\psi_*(y)$$

which changing $a_1 + (\lambda_3 y - 2\lambda_1)/2(y-c)$ to a_1,

$$= \sum_{a_1, y\neq c} \sigma\left((y-c)a_1^2 + \frac{R(y)}{y-c}\right)\psi_*(y)$$

which using $\sum_{a_1} \sigma(va_1^2 + w) = \sum_{t\neq 0} \sigma(v)\psi_*(tw)$ for $v \neq 0$,

$$= \sum_{y\neq c, t\neq 0} \sigma(y-c)\psi_*\left(\frac{tR(y)}{y-c}\right)\psi_*(y)$$
$$= \sum_{y\neq c, t} \sigma(y-c)\psi_*\left(\frac{tR(y)}{y-c}\right)\psi_*(y) + \sum_{y\neq c} \sigma(y-c)\psi_*(y)$$

which since μ is the unique root of $R(y)$,

$$= q\sigma(\mu - c)\psi_*(\mu) - \sum_u \psi_*(u^2+c).$$

We need to show that if $\mu \neq c$, then $\sigma(\mu - c) = -1$. Let

$$P(T) = (T^2 + \zeta T + \xi)(T^2 + \zeta^q T + \xi^q)$$

be the factorization of $P(T)$ over R_E. It is easy to see $\mu - c = ((\zeta - \zeta^q)/2)^2$. Thus if $\mu - c$ is a square in F_*, $\zeta - \zeta^q$ is in F_*. Since $\zeta + \zeta^q = \lambda_3$ which is in F_*, $\zeta = \zeta^q$. Thus μ must be equal to c.

Next, we prove (2.9.2). By a simple polynomial manipulation, we obtain, $R(c) = -((\lambda_3 c - 2\lambda_1)/2)^2$. Since μ is the unique root of $R(y)$, $\lambda_3 c - 2\lambda_1 = 0$ if and only if $\mu = c$. Thus

$$S_2 = \begin{cases} q\sigma(\mu^2 - 4\lambda_0)\psi_*(\mu) & \text{if } \mu = c, \\ 0 & \text{otherwise.} \end{cases}$$

We have to show $\sigma(\mu^2 - 4\lambda_0) = -1$ if $\mu = c$. Since $\mu^2 - 4\lambda_0 = (\xi - \xi^q)^2$, if $\mu^2 - 4\lambda_0$ is a square in F_*, ξ is in F_*. Since $\mu = c$, ζ is also in F_*. This contradicts the irreducibility of $P(T)$ over R_F. □

Finally, we can evaluate S_P for irreducible polynomials of degree 4.

Theorem 2.10. *Let $P(T) = T^4 + \lambda_3 T^3 + \lambda_2 T^2 + \lambda_1 T + \lambda_0$ be an irreducible polynomial over R_F, and $P(T) = P_1(T)P_2(T)$ be the factorization of $P(T)$ over R_E. Then,*

$$S_P = q^2(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

Hence,

$$\lambda_P = \nu_*(P_1(0)) + \nu_*(P_2(0)).$$

Proof. Using Lemmas 2.6 and 2.7,

$$S_P = q\sum_{a_0} \chi_*(M_1) + q\sum_{a_1^2 \neq a_0} \chi_*(M_2)$$

where

$$M_1 = \begin{bmatrix} & 1 \\ 1 & a_0^2 \end{bmatrix}\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & -(\lambda_2 + a_0\lambda_3) \\ & 1 \end{bmatrix}\begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} & 1 \\ 1 & -a_1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & (a_1^3 - 2a_0a_1)/(a_1^2 - a_0)^2 \end{bmatrix}\begin{bmatrix} 1/(a_1^2 - a_0) & \\ & a_1^2 - a_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -(\lambda_1 + a_1\lambda_2 + a_0\lambda_3) \\ & 1 \end{bmatrix}\begin{bmatrix} \lambda_0 & \\ & 1 \end{bmatrix}.$$

It is easy to see that $\det(M_1) = \lambda_0$, $\mathrm{tr}(M_1) = a_0^2 + \lambda_3 a_0 + \lambda_2$, and M_1 cannot be the identity. Thus,

$$\sum_{a_0} \chi_*(M_1) = \sum_{a_0} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & a_0^2 + \lambda_3 a_0 + \lambda_2 \\ & 1 \end{bmatrix}\right) = \sum_{u} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & u^2 + c \\ & 1 \end{bmatrix}\right).$$

On the other hand, $\det(M_2) = \lambda_0$,

$$\mathrm{tr}(M_2) = (a_0^2 + \lambda_0 + a_1(\lambda_3 a_0 + \lambda_2 a_1 + \lambda_1))/(a_1^2 - a_0),$$

and M_2 becomes the identity if and only if $a_1 = 0$, $a_0^2 = \lambda_0$, and $\lambda_3 a_0 + \lambda_1 = 0$. Let $I = \{a_0 : a_0^2 = \lambda_0, \lambda_3 a_0 + \lambda_1 = 0\}$. Then,

$$\sum_{a_1^2 \neq a_0} \chi_*(M_2) = \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right) + \sum_{a_1^2 \neq a_0} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & (a_0^2 + \lambda_0 + a_1(\lambda_3 a_0 + \lambda_2 a_1 + \lambda_1))/(a_1^2 - a_0) \\ & 1 \end{bmatrix}\right) - \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & (a_0^2 + \lambda_0)/ - a_0 \\ & 1 \end{bmatrix}\right)$$

which, by Lemma 2.9 and by the Fourier inversion formula,

$$= \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right) - q\chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \mu \\ & 1 \end{bmatrix}\right)$$
$$- \sum_{u} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & u^2 + c \\ & 1 \end{bmatrix}\right)$$
$$- \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & (a_0^2 + \lambda_0)/ - a_0 \\ & 1 \end{bmatrix}\right)$$

Thus

$$S_P = q \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right) - q^2\chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \mu \\ & 1 \end{bmatrix}\right)$$
$$- q \sum_{a_0 \in I} \chi_*\left(\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & (a_0^2 + \lambda_0)/ - a_0 \\ & 1 \end{bmatrix}\right)$$

It is easy to show that I contains exactly one element if $P_1(0)$ is in F_*, and that I is empty if $P_1(0)$ is not in F_*. If $P_1(0)$ is in F_*, then both

$$\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \mu \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & (a_0^2 + \lambda_0)/ - a_0 \\ & 1 \end{bmatrix}$$

are parabolic. Hence, using the known values of the character χ_*,

$$S_P = q(q-1) - q^2(-1) - q(-1) = 2q^2 = q^2(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

The last equality is obtained because ν_* is trivial on F_*. On the other hand if $P_1(0)$ is not in F_*, then

$$\begin{bmatrix} & -\lambda_0 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \mu \\ & 1 \end{bmatrix}$$

is elliptic. Hence,

$$S_P = -q^2(-\nu_*(P_1(0)) - \nu_*(P_2(0))) = q^2(\nu_*(P_1(0)) + \nu_*(P_2(0))).$$

□

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